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ŚRĪYANTRA - A STUDY OF SPHERICAL AND PLANE FORMS

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Śrīyantra is an ancient geometrical form with great traditional signficance and current interest in scientific studies and architectural applications. It consists of a complex network of intersecting triangles in its central region. The construction of the plane figure from traditional literature is adequate for small objects for worship. Errors at intersections of certain lines becomes significant for large sizes of the figure. These erorrs occur due to the incompatibility of the redundant measurements in the specifications. Accurate measurements from recent literature are available for specific configurations of plane form. This paper presents a comprehensive study of the geometrical properties of the triangular network of spherical and plane forms of *Śrīyantra*.

A general formulation of the triangular network for the spherical form is presented using spherical trigonometry. Geometrical relationship observed in specimens of plane figures from different sources are expressed as mathematical equations. The resulting system of simultaneous nonlinear equations is solved by numerical methods for each set of selected relationships for obtaining the parameters required for drawing the figure.

Figures with closest resemblance to the traditional figure are obtained by non-linear optimization. The formulation for the plane form is derived as a simplification of the formulation of spherical form. The figure of *Śrīyantra* of both plane and spherical forms can be constructed to any required size ranging from miniature objects of worship to temple structures with a high degree of accuracy from the results of this study.

Keywords : *Kurma-prishtha, Meru,* Non-linear optimization. Numerical methods, Simultaneous non-linear equations, Spherical trigonometry, *Śrīyantra, Śrīcakra.*

Introduction

The development of various architectural forms through centuries of history is well known. Definite shapes and patterns of pyramids, Churches, Monuments, Temples and also Modern structures can be distinctly recognised. In recent structures, the *Meru* temple in Devipuram near Visakhapatnam in Andhra Pradesh uses *Śrīyantra* as the plan view for its multi-storeyed structure.

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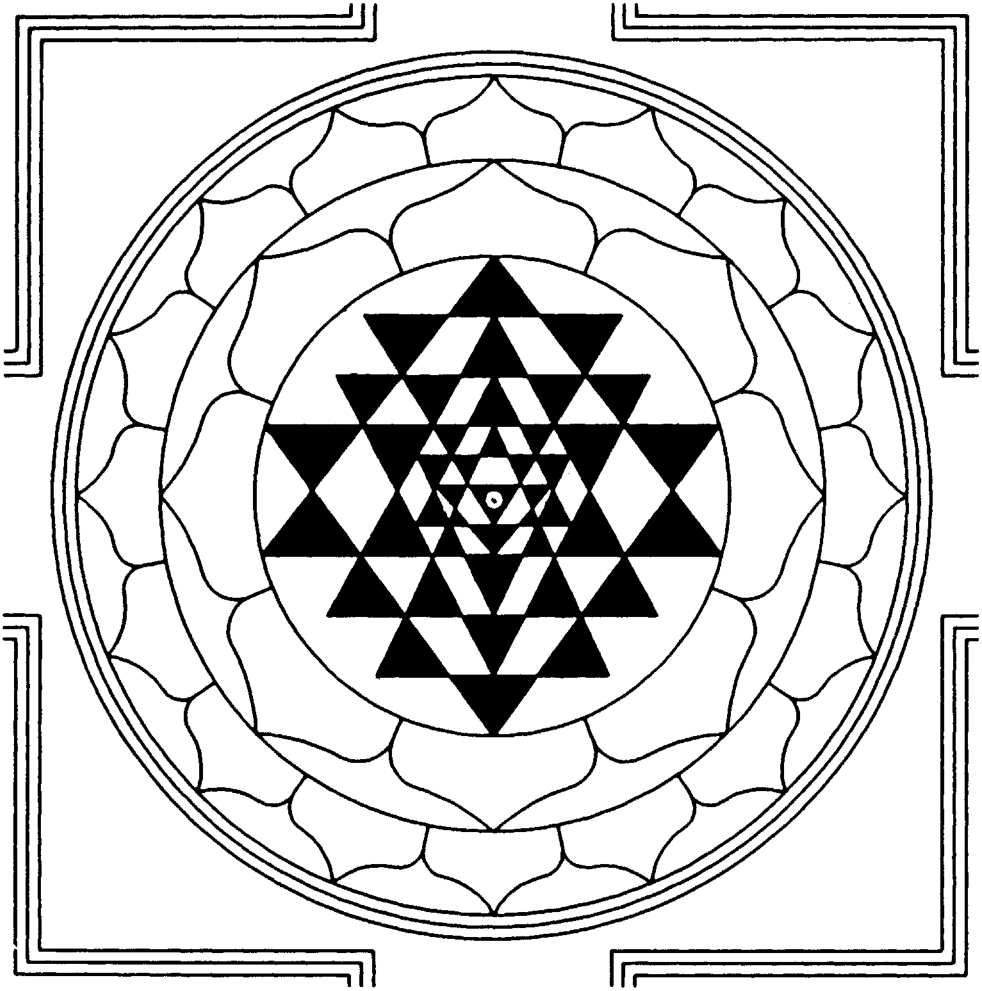


Fig. 1 *Śrīyantra*—The Tantra Figure

The manifestation of the world process, the experiences of the universe and the individual are explained in the introduction of *Sivasutras.*[1] The essence of the universe is *Siva*. His power of action is *Sakti*. The essence in every individual is is called *Jiva*. *Siva* is unlimited in all his attributes whereas *Jiva* is limited in all respects.

The central point in the *Śrīyantra* called *Bindu* is identified with the unified *Siva* and *Sakti*. The whole of *Śrīyantra* shown in Fig. 1 considered as the expanded form of the *Bindu* consists of 9 regions called enclosures. The Inner Sector is a triangle complex consisting of a network of triangles circumscribed by a circle. It is composed of five primary triangles called *Sakti* triangles with vertices downward and four primary triangles called *Siva* triangles with vertices upward. These triangles have parallel bases with symmetric left and right sides. The mutual intersection of these 9 primary triangles results in the formation of 43 secondary triangles.

The central point of the figure is considered as the first enclosure. The innermost secondary triangle is counted as second. Enclosures from third to sixth consist of 8, 10, 10 and 14 secondary triangles respectively. The outer sector consists of 3 enclosures. The 7th enclosure consists of an eight petalled lotus around the circum circle of the triangle complex and lying within a second circum circle. The 8th enclosure consists of a sixteen petalled lotus around the second circum circle and lying within a third circum circle. A triplet of closely spaced circles are used in the place of the third circum circle. The 9th enclosure consists of a set of three concentric squares outside the outermost circum circle. Openings on all four sides of the squares called gateways are provided.

A popular description and construction of the complete plane figure having its origin from tantra literature is given in the commentary on *Saundarya Lahari*.[2] The triangle complex consists of spacings of 6, 6, 5, 3, 3, 4, 3, 6, 6, 6 units of distance between successive parallel base lines of the primary triangles. The radius of the circum circle is 24 units. This figure corresoponding to the above construction is called tantra figure.

There are three major versions of the yantra in popular use. The plane form contains all the enclosures in a single plane where triangles and squares are formed by straight lines. In the *Meru* form, different enclosures lie in different horizontal planes at different elevations. The square at outermost enclosure is at the base level and successive enclosures are at progressively higher levels with the central point placed at the highest elevation.

In the spherical form, the triangular complex of the first 6 enclosures is formed by spherical triangles on a spherical surface. The sides of the triangles are taken as arcs of great circles on the spherical surface. The outer sector is placed either in a single plane or in a stepped form. This form is known as *Kurma-prishtha Meru*. Photographs of *Śrīyantra* of spherical form made of rock crystals are given by Madhu Khanna.[3] A figure of *Śrīyantra* on a copper plate with elliptical arcs as sides of the triangles is given by Ajit Mookerjee.[4]

The significance of *Śrīyantra* as an abode of the presiding deities of the cosmic organisation is indicated in *Atharva veda*.[5] A detailed exposition of these principles correlating these powers in the cosmic organisation and the organisation within a human individual is given in *Bhavanopanishat*.[6] A symbolic representation of the hierarchical principal and powers operating in the organisation of the cosmos in *Śrīyantra* based on a thorough study of various *Tantras* is given by Sir John Woodroff.[7]

One of the greatest applications of *Śrīyantra* is the traditional worship[8] where the goal is to realise the identification of the individual with entire universe. A treatment of procedures of the worship and their significance is given in *Srividya Saparya Vasana*.[9]

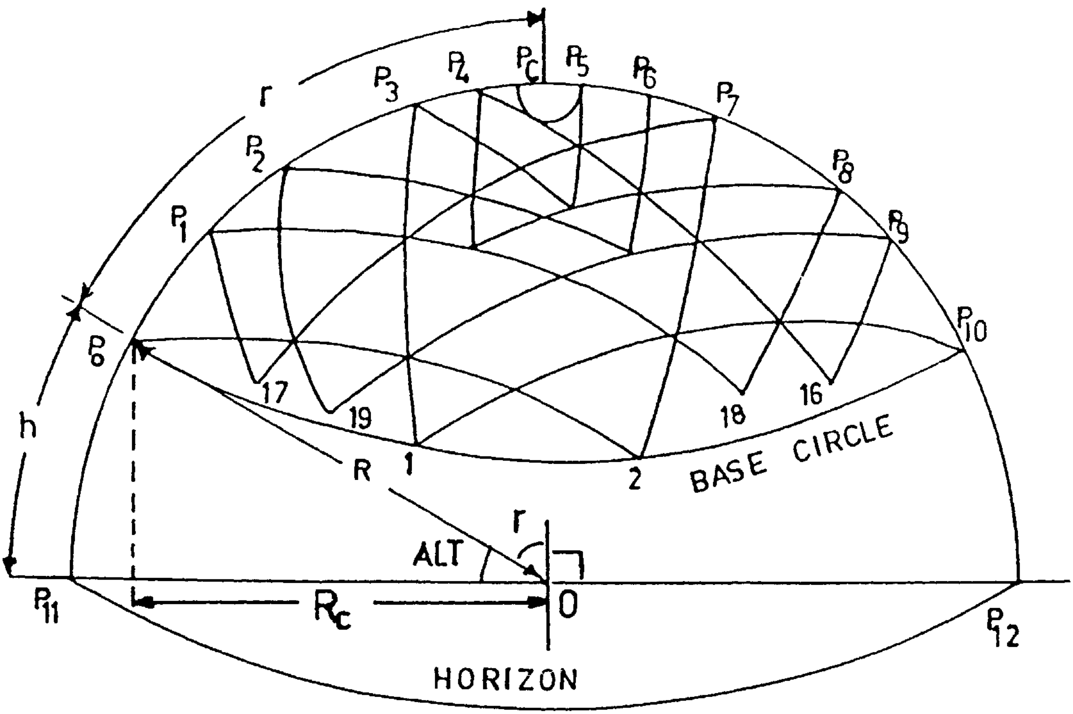


Fig. 2.1 : Spherical Triangular Complex

Kulaichev and Ramendic[10] investigated the visual influence of *Śrīyantra* on the psycho-physiological states of man. They studied the influence of visual stimulus of gazing at the figure of *Śrīyantra* on the response of Electroencephalograms of several individuals. They concluded that *Śrīyantra* is a unique composition to concentrate on for achieving certain states of consciousness. Raj Bapna[11] states that the original purpose of *Śrīyantra* was to develop mind power. It is stated that gazing at the central dot of the *Śrīyantra* increases the communication between left brain and right brain by which the power of concentration is enhanced.

The errors of intersection of the lines in the tantra figure are shown by Kulaichev.[12] He gives a mathematical formulation for drawing the triangle complex of a plane figure. A set of 5 geometric properties were used. The required parameters were obtained by solving a set of non-linear algebraic equations by nested iterative procedures. Diagrams of spherical form of *Śrīyantra* were given without any mathematical formulations.

CONSTRUCTION OF SPHERICAL FORM

The formulation of the geometry of construction is based on spherical trigonometry. The spherical triangular complex is shown in Fig. 2.1. The upper part of the hemisphere above horizontal plane is shown. The radius of the sphere is R with centre at point O. The vertical circle arc containing the points P_{11}, P_c and P_{12} forms the arc of symmetry of the triangular complex. The centre of the triangular complex P_c is vertically above the centre of the sphere. The horizontal base circle of radius R_c passing through points P_0-1-2-P_{10} forms the circum circle around the triangular complex. The great circle arc "r" is the angle subtended by the arc P_0P_c or P_cP_{10} at the centre of the sphere. The altitude "h" is angle subtended by arc $P_{11}P_{10}$ or $P_{12}P_{10}$ at the centre of the sphere. These parameters can be ralated as follows.

$$r + h = \frac{\pi}{2} \quad \text{.....(2.1a)}$$

$$Rc = R\cos(h) = R\sin(r) \quad \text{.....(2.1)}$$

A spherical triangle is formed by three arcs of great circles intersecting at three points. A side is denoted by the angle subtended by the great circle arc at the centre of the sphere. The length of the side is obtained by the product of the angle in radians and the radius in linear units. The angle at the point of intersection of two sides is taken as the angle between the planes of the two great circle arcs forming the sides. The points P_0 to P_4 and P_7 to P_{10} form the vertices of 9 root triangles. The base lines of the root triangles intersect the arc of symmetry at right angles at the base points P_1 to P_9.

The variables in the spherical triangle complex are shown in Fig 2.2. The intercepts g, c and b mark the base points P_4, P_3 and P_1 with respect to the centre P_c. on the arc of symmetry. The intercepts d and e locate the base points P_7 and P_9. The remaining base points P_2, P_5, P_6, and P_8 are obtained during the construction. The intercepts a and f are related to these as follows.

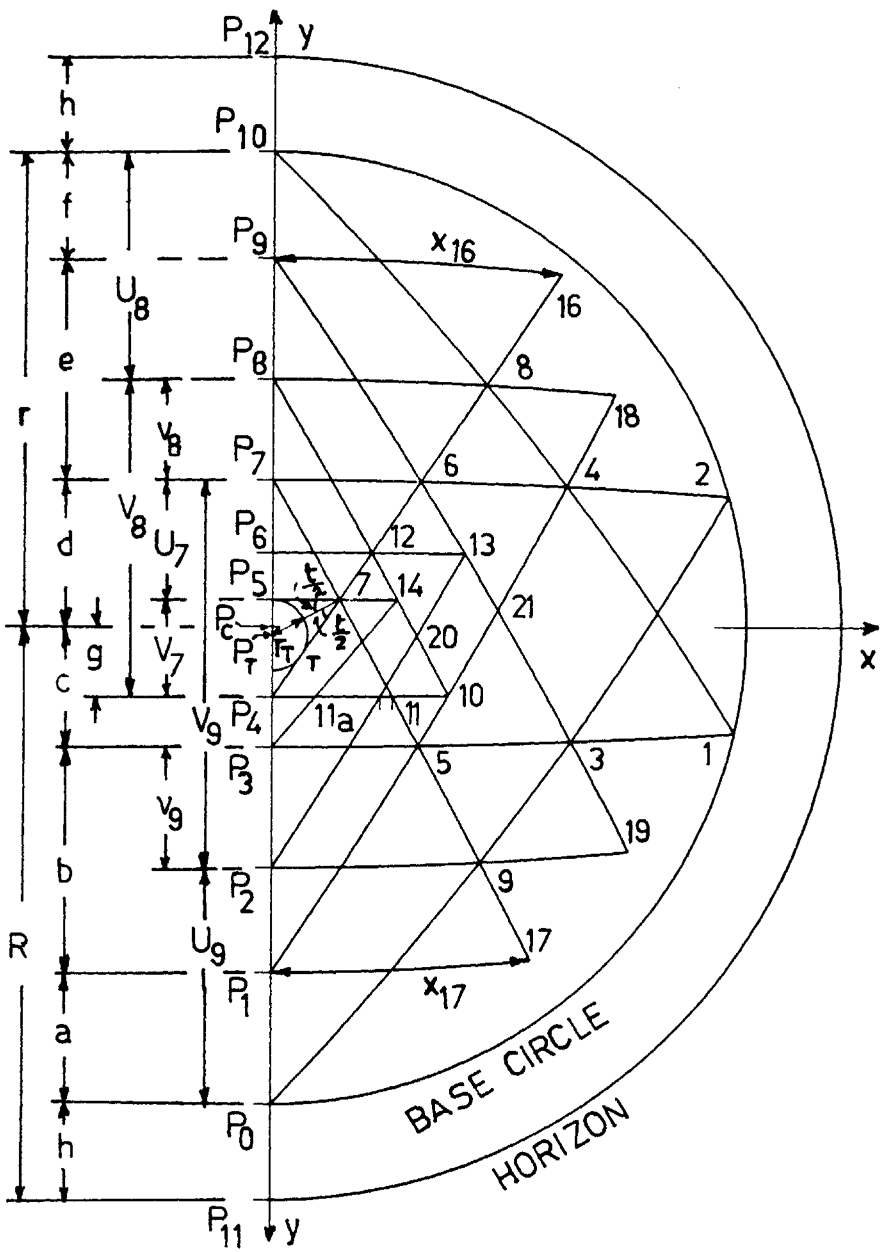


Fig. 2.2 Spherical Form : Points and Variables

$$r = a + b + c = d + e + f = \frac{\pi}{2} - h \qquad (2.2)$$

The 6 independent variables b, c, d, e, g and h in radians are chosen as the basic variables in the formulation. The sequence of construction of the figure is described along with the expressions for several intermediate variables like x_i, U_i as follows.

The base great circle arc $P_3 1$ perperpendicular to arc of symmetry $P_{11}P_{12}$ at P_3 meets the circumscribing circle at point 1. The length of arc $P_3 1$ is denoted by x_1. The lengths of other base lines are similarly denoted by x_n where n represents the number denoting the corner of the triangle.

In the right angled spherical triangle $P_c P_3 1$

arc $P_3 1 = x_1$ arc $P_C 1 = r = \frac{\pi}{2} - h$

arc $P_3 P_c = c$ angle $P_c P_3 1 = \frac{\pi}{2}$

The arc x_1 can be obtained from Napier's rules[13] for right angled triangles as follows:

$$\text{Cos } x_1 = \frac{\text{Cos } r}{\text{Cos } c} \qquad (2.3)$$

Similarly the base great circle arc drawn perpendicular to arc of symmetry at P_7 meets the circumscribing circle at point 2. The arc length $P_7 2 = x_2$ is obtained from the spherical triangle $P_c P_7 2$ right angled at P_7 as

$$\text{Cos } x_2 = \frac{\text{Cos } r}{\text{Cos } d} \qquad (2.4)$$

Point 3 is the intersection of the transverse line $P_0 2$ with the base line $P_3 1$. Solution of the spherical triangles $P_0 P_3 3$ and $P_0 P_7 2$ right angled at P_3 and P_7 respectively and having a common angle at vertex P_0 gives

$$\sin(r - c) = \tan(x_3) \cot(P_0) \text{ and } \sin(r + d) = \tan(x_2) \cot(P_0) \qquad (2.5a)$$

The arc $P_3 3 = x_3$ is obtained by eliminating P_0 from eq(2.5a)

$$\tan x_3 = \frac{\text{Sin}(r - c)}{\text{Sin}(r + d)} \tan x_2 \qquad (2.5)$$

Point 4 is the intersection of the transverse line $P_{10} 1$ with the base line $P_7 2$. Arc length $P_7 4 = x_4$ is similarly obtained from triangles $P_{10} P_7 4$ and $P_{10} P_3 1$ as

$$\tan x_4 = \frac{\text{Sin}(r - d)}{\text{Sin}(r + c)} \tan x_1 \qquad (2.6)$$

Point 5 is obtained by the intersection of transverse line $P_1 4$ with the base line $P_3 1$. Arc length $P_3 5 = x_5$ is obtained from the triangles $P_1 P_3 5$ and $P_1 P_7 4$ as

$$\tan x_5 = \frac{\sin (b)}{\sin (b + c + d)} \tan x_4 \qquad (2.7)$$

Point 6 is obtained by the intersection of the transverse line $P_9 3$ with base line $P_7 2$. The arc length $P_7 6 = x_6$ is obtained similar to x_5 as

$$\tan x_6 = \frac{\sin (e)}{\sin (c + d + e)} \tan x_3 \qquad (2.8)$$

Point 7 is obtained by the intersection of the two transverse arcs $P_7 5$ and $P_4 6$. The base point P_5 is located by drawing the great circle arc through point 7 which meets the arc of symmetry at right angles. The arc length $P_5 7 = x_7$ is obtained as follows.

Let S_7 be the arc length between P_4 and P_7. The Point P_5 lies between P_4 and P_7 such that $P_5 P_7 = U_7$ and $P_4 P_5 = V_7$. It can be seen from the Fig. 2.2 that

$$S_7 = U_7 + V_7 = d + g \qquad (2.9)$$

Use of the relations similar to eq (2.5a) for the right angled triangles $P_3 P_7 5$ and $P_5 P_7 7$ with common vertex at P_7 gives

$$\frac{\sin U_7}{\sin (c + d)} = \frac{\tan x_7}{\tan x_5} \qquad (2.10)$$

Similarly from the right angled triangles $P_7 P_4 6$ and $P_5 P_4 7$ with P_4 as common vertex

$$\frac{\sin V_7}{\sin (d + g)} = \frac{\tan x_7}{\tan x_6} \qquad (2.11)$$

The quotient of the two unknowns U_7 and V_7 is obtained by eliminating the required unknown x_7 from eq (2.10 and 2.11) as

$$Q_7 = \frac{\sin V_7}{\sin U_7} = \frac{\sin (d + g)}{\sin (c + d)} \frac{\tan x_5}{\tan x_6} \qquad (2.12)$$

Substitution of $V_7 = (S_7 - U_7)$ from eq (2.9) in first part of eq (2.12) gives

$$\tan U_7 = \frac{\sin S_7}{Q_1 + \cos S_7} \qquad (2.13)$$

x_7 can be obtained from eq (2.11) as

$$\tan x_7 = \frac{\sin U_7}{\sin(c+d)} \tan x_5 \qquad (2.14)$$

Point 8 is obtained similarly by the intersection of the two transverse arcs $P_4 6$ extended and $P_{10} 1$. The base point P_8 is located by drawing the great circle arc through point 8 to meet the arc of symmetry at right angles. The following relations based on point 8 can be obtained similar to eq(2.9 to 2.14).

$$S_8 = U_8 + V_8 = r + g \qquad (2.15)$$

$$\frac{\sin U_8}{\sin(r+c)} = \frac{\tan x_8}{\tan x_1} \qquad (2.16)$$

$$\frac{\sin V_8}{\sin(d+g)} = \frac{\tan x_8}{\tan x_6} \qquad (2.17)$$

$$Q_7 = \frac{\sin V_8}{\sin U_8} = \frac{\sin(d+g)}{\sin(r+c)} \frac{\tan x_1}{\tan x_6} \qquad (2.18)$$

$$\tan U_8 = \frac{\sin S_8}{Q_8 + \cos S_8} \qquad (2.19)$$

The arc length $P_8 8 = x_8$ can be obtained from eq(2.16) as

$$\tan x_8 = \frac{\sin U_8}{\sin(r+c)} \tan x_1 \qquad (2.20)$$

$$v_8 = r - U_8 - d \qquad (2.21)$$

Point 16 is obtained by the intersection of the base line through P_9 with the transverse line $P_4 6$ extended. The base arc length $P_9 16 = x_{16}$ is obtained from triangles with common vertex at P_4 as

$$\tan x_{16} = \frac{\sin(d+e+g)}{\sin(r+c)} \tan x_6 \qquad (2.22)$$

The transverse line $P_7 5$ gives the points 11 and 17 by intersection with base lines through P_4 and P_1 respectively. The base arcs $P_4 11 = x_{11}$ and $P_1 17 = x_{17}$ are obtained as

$$\tan x_{11} = \frac{\sin(d+g)}{\sin(c+d)} \tan x_5 \qquad (2.23)$$

$$\tan x_{17} = \frac{\sin(b+c+d)}{\sin(c+d)} \tan x_5 \qquad (2.24)$$

The same transverse line P_75 intersects another transverse line P_02 to locate the point 9. The following relations for point 9 can be derived similar to eq (2.9 to 2.14)

$$S_9 = U_9 + V_9 = r + d \qquad (2.25)$$

$$\frac{\sin U_9}{\sin(r+d)} = \frac{\tan x_9}{\tan x_2} \qquad (2.26)$$

$$\frac{\sin V_9}{\sin(c+d)} = \frac{\tan x_9}{\tan x_5} \qquad (2.27)$$

$$Q_9 = \frac{\sin V_9}{\sin U_9} = \frac{\sin(c+d)}{\sin(r+d)} \frac{\tan x_2}{\tan x_5} \qquad (2.28)$$

$$\tan U_9 = \frac{\sin S_9}{Q_9 + \cos S_9} \qquad (2.29)$$

The arc length $P_29 = x_9$ can be obtained from eq(2.26) as

$$\tan x_9 = \frac{\sin U_9}{\sin(r+d)} \tan x_2 \qquad (2.30)$$

$$v_9 = r - U_9 - c \qquad (2.31)$$

Point 10 is ovtained by extending the base line through P_4 to meet the transverse line P_14 which gives the arc length $P_410 = x_{10}$ as

$$\tan x_{10} = \frac{\sin(b+c-g)}{\sin(b+c+d)} \tan x_4 \qquad (2.32)$$

Point 18 is intersection of the base line through P_8 with the transverse line P_14 produced which gives the arc length $P_818 = x_{18}$ as

$$\tan x_{18} = \frac{\sin(b+c+d+v_8)}{\sin(b+c+d)} \tan x_4 \qquad (2.33)$$

Point 12 is obtained by the intersection of the transverse arcs P_46 and P_810. The great circle arc through point 12 is drawn to meets the arc of symmetry at right angles at the base point P_6. The relations based on point 12 similar to eq (2.9 to 2.14) arc.

$$S_{12} = U_{12} + V_{12} = d + g + v_8 \tag{2.34}$$

$$\frac{\sin U_{12}}{\sin (d + g)} = \frac{\tan x_{12}}{\tan x_6} \tag{2.35}$$

$$\frac{\sin V_{12}}{\sin (d + g + v_8)} = \frac{\tan x_{12}}{\tan x_{10}} \tag{2.36}$$

$$Q_{12} = \frac{\sin V_{12}}{\sin U_{12}} = \frac{\sin (d + g + v_8)}{\sin (d + g)} \frac{\tan x_6}{\tan x_{10}} \tag{2.37}$$

$$\tan U_{12} = \frac{\sin S_{12}}{Q_{12} + \cos S_{12}} \tag{2.38}$$

Arc length $P_6 12 = x_{12}$ can be obtained from eq (2.35) as

$$\tan x_{12} = \frac{\sin U_{12}}{\sin (d + g)} \tan x_6 \tag{2.39}$$

$$v_{12} = d + g - U_{12} \tag{2.40}$$

Point 14 is the intersection of the base line through P_5 with the transverse line $P_8 10$ so that $P_5 14 = x_{14}$. The transverse arc $P_3 14$ is drawn.

$$\tan x_{14} = \frac{\sin (U_7 + v_8)}{\sin (d + g + v_8)} \tan x_{10} \tag{2.41}$$

Points 13 and 19 are located by the intersection of transverse line $P_9 3$ with base lines through P_6 and P_2 respectively so that $P_6 13 = x_{13}$ and $P_2 19$ -= x_{19}.

$$\tan x_{13} = \frac{\sin (e + v_{12})}{\sin (c + d + e)} \tan x_3 \tag{2.42}$$

$$\tan x_{19} = \frac{\sin (c + d + e + v_9)}{\sin (c + d + e)} \tan x_3 \tag{2.43}$$

The arc $P_2 13$ only remains to be drawn. This line should pass through point 11 which was already obtained by intersection of lines $P_7 5$ and $P_4 10$. Let the point of intersection of arcs $P_2 13$ and $P_4 10$ be at 11_a so that $P_4 11_a = x_{11a}$.

$$\tan x_{11a} = \frac{\sin (v_9 + c - g)}{\sin (v_9 + c + d - v_{12})} \tan x_{13} \tag{2.44}$$

FORMULATION OF CONSTRAINTS OF FIGURE

Some of the geometrical relationships observed in available specimens of *Śrīyantra* are formulated as constraint functions. The value of the function defined must be equal to zero for the relationship to be satisfied.

1. Because of the sequence of construction adapted for drawing a third line through the earlier intersection point of a pair of lines, all intersections will be perfect except at point 11. The essential condition of concurrency at all points to be satisfied is that the two points of intersection 11 and 11_a must be at the same location. This requires that the six intercepts b, c, d, e, g and h must be compatible such that $x_{11} = x_{11a}$. This may be stated as the error of intersection F_1 defined as follows must be zero.

$$F_1 = x_{11} - x_{11a} \qquad (3.1)$$

2. The inscribed circle of the innermost secondary triangle viewed as a magnification of the centre point must be concentric with the circumscribing circle around the triangular complex. The distance between the centres of the two circles defined as function F_4 represents the error in concentricity of the two circles. The centre P_T of the inscribed circle of radius r_T is at the intersection of the arc of symmetry P_4P_5 and the bisector of the angle P_47P_5. The transverse arc P_47 and the base arc P_57 are tangential to the inscribed circle at points T and P_5 respectively (Fig. 2.2).

In the spherical triangle P_4P_57 right anged at P_5

angle $P_5\,7\,P_4 = t$ and $P_4P_5 = V_7$

$$\sin x_7 = \tan V_7 \tan\left(\frac{\pi}{2} - t\right) = \frac{\tan(d + g - U_7)}{\tan t} \qquad (3.2a)$$

$$\tan t = \frac{\tan(d + g - U_7)}{\sin x_7} \qquad (3.2b)$$

In the spherical triangle P_TP_57 right angled at P_5

angle $P_5\,7P_T = \frac{t}{2}$ and $P_T\,P_5 = r_T$

$$\tan r_T = \sin(x_7)\tan\left(\frac{t}{2}\right) \qquad (3.2c)$$

The concentricity error F_2 defined as the arc P_CP_T is

$$F_2 = d - U_7 - r_T \qquad (3.2)$$

3. The root triangle with vertex at P_8 is equilateral. Solving the triangle P_8P_410 with the arc length $P_810 = 2\,x_{10}$ gives

$$\cos V_8 = \frac{\cos(2x_{10})}{\cos(x_{10})} \qquad (3.3a)$$

$$F_3 = \cos(d + g + V_8) - \frac{\cos(2x_{10})}{\cos(x_{10})} \qquad (3.3)$$

4. The root triangle with vertex at P_2 is equilateral. The condition function is obtained similar to eq (2.48) as.

$$F_4 = \cos(c + d + g + v_9 - v_{12}) - \frac{\cos(2x_{13})}{\cos(x_{13})} \qquad (3.4)$$

5. The base arcs $P_6 13$ and $P_4 10$ are equal.

$$F_5 = x_{10} - x_{13} \qquad (3.5)$$

6. The inner most secondary traingle with vertex at P_4 is equilateral. The function is obtained similar to eq (2.48) as.

$$F_6 = \cos(d + g - U_7) - \frac{\cos(2x_7)}{\cos(x_7)} \qquad (3.6)$$

7. The base arcs $P_8 18$ and $P_2 19$ are equal.

$$F_7 = x_{18} - x_{19} \qquad (3.7)$$

8. Point 16 lies on the circumscribing circle of radius r.

$$F_8 = r - r_{16} \qquad (3.8)$$

where r_{16} is the arc $P_C 16$ of tbe triangle $P_C P_9 16$ right angled at P_9

$$\cos(r_{16}) = \cos(d + e)\cos(x_{16}) \qquad (3.8a)$$

9. Point 17 lies on the circumscribing circle.

$$F_9 = r - r_{17}$$

where r_{17} is the arc $P_C 17$ of the triangle $P_C P_1 17$ right angled at P_1

$$\cos(r_{17}) = \cos(b + c)\cos(x_{17}) \qquad (3.9a)$$

10. The intercepts P_1P_4 and P_4P_8 are equal.

$$F_{10} = b + c - d - 2g - v_8 \qquad (3.10)$$

11. The intercepts P_9P_6 and P_6P_2 are equal.

$$F_{11} = c + d + v_9 - 2v_{12} - e \qquad (3.11)$$

12. The base arc lengths $P_9 16$ and $P_1 17$ are equal.

$$F_{12} = x_{16} - x_{17} \tag{3.12}$$

13. The transverse arcs $P_8 10$ and $P_2 13$ intersect at point 20. The base line passing through 20 intersects the arc of symmetry at P_{20}. The condition is that P_5 is the mid point of the arc $P_6 P_{20}$. By using relations similar to eq (2.9 to 2.14) we obtain

$$F_{13} = U_7 - \frac{(U_{20} - v_8 + v_{12})}{2} \tag{3.13}$$

$$\text{where } \tan U_{20} = \frac{\sin S_{20}}{Q_{20} + \cos S_{20}} \tag{3.13a}$$

$$Q_{20} = \frac{\sin(c + d + v_9 - v_{12})}{\sin(d + g + v_8)} \frac{\tan x_{10}}{\tan x_{13}} \tag{3.13b}$$

$$\text{and } S_{20} = c + d + v_8 + v_9 \tag{3.13c}$$

14. The transverse arcs $P_9 19$ and $P_1 18$ intersect at point 21. The base line passing through 21 intersects the arc of symmetry at P_{21}. The condition is that P_6 is the mid point of the arc $P_7 P_{21}$. Similar to condition 13, one can obtain

$$F_{14} = v_{12} - \frac{(U_{21} - e)}{2} \tag{3.14}$$

$$\text{where } \tan U_{21} = \frac{\sin S_{21}}{Q_{21} + \cos S_{21}} \tag{3.14a}$$

$$Q_{21} = \frac{\sin(b + c + d)}{\sin(c + d + e)} \frac{\tan x_{10}}{\tan x_{13}} \tag{3.14b}$$

$$\text{and } S_{21} = b + c + d + e \tag{3.14c}$$

15. The condition is similar to the previous one. The condition here is that P_4 is the mid point of the arc $P_3 P_{21}$. Similar to condition 14, we obtain

$$F_{15} = g + \frac{(d + e + - c - U_{21})}{2} \tag{3.15}$$

16. Points 16 and 17 are equidistant from the centre of the figure P_C

$$F_{16} = r_{16} - r_{17} \tag{3.16}$$

17. Points 18 and 19 are equidistant from the centre of the figure P_C.

$$F_{17} = r_{18} - r_{19} \quad (3.17)$$

where the radial distances r_{18} and r_{19} derived similar to eq (3.8a) are.

$$\cos(r_{18}) = \cos(d + y_8)\cos(x_{18}) \quad (3.18a)$$

$$\cos(r_{19}) = \cos(c + y_9)\cos(x_{19}) \quad (3.18b)$$

18. Points 16 and 18 are equidistant from the centre of the figure P_C.

$$F_{18} = r_{16} - r_{18} \quad (3.18)$$

19. Points 17 and 19 are equidistant from the centre of the figure P_C.

$$F_{19} = r_{17} - r_{19} \quad (3.19)$$

20. The outer most root triangles with vertices at P_0 and P_{10} are identical.

$$F_{20} = c - d \quad (3.20)$$

CONSTRAINED FIGURES

For generating the triangle complex on a spherical surface of radius R, the 6 basic variables b, c, d, e, g and h are required. The other parameters required can be found from the formulation of the construction. Each basic variable may be assigned any arbitrary value or may be fixed according to some specified geometrical property. Since there are 6 basic variables, any 6 constraints can be satisfied at a time in a figure. Among the constraints formulated, the concurrency condition in eq (3.1) and the concentricity condition in eq (3.2) are considered essential. Any other 4 constraints from F_3 to F_{20} may be used along with the 2 essential conditions to form a set of conditions.

The problem is to find the values of a set of basic variables which would give a figure satisfying the selected set of conditions. This requires that the 6 selected constraint functions must be equal to zero simultaneously. Newton-Raphson[14] method is used for solving this system of simultaneous non-linear equations. A set of approximate initial values are selected for the basic variables. The numerical values of the 6 selected constraint functions are computed using these initial values. The partial derivatives of these functions with respect to the basic variables are numerically evaluated by forward finite difference approximations. Corrections to be applied to the initial values are obtained by solving the resulting system of linear simultaneous equations. The values of the 6 constraint functions are computed using the corrected values of the basic variables. If any one of the values is greater than a specified tolerance value, the corrected values of the basic variables are taken as new initial values and new corrections are found. The iterations are continued till the value of each of the 6 functions is less than the tolerance.

TABLE 1. BASIC VARIABLES OF SPHERICAL CONSTRAINED FIGURES (RADIANS)

Constraints	b	c	d	e	g	h
1, 2, 3, 5, 10, 19	0.105036	0.054376	0.065419	0.105517	0.024275	1.344437
1, 2, 3, 6, 16, 19	0.450462	0.442391	0.445729	0.425478	0.183904	0.539209
1, 2, 3, 10, 19, 20	0.244730	0.158369	0.158369	0.225411	0.060166	1.042990
1, 2, 4, 5, 10, 19	0.231687	0.120012	0.146680	0.230471	0.053009	1.076084
1, 2, 4, 8, 13, 19	0.463973	0.753761	0.890177	0.375039	0.466743	0.261736
1, 2, 4, 10, 15, 19	0.252893	0.132925	0.160863	0.249384	0.057987	1.031951
1, 2, 6, 8, 9, 10	0.449590	0.332617	0.328263	0.430650	0.129804	0.722398
1, 2, 8, 9, 16, 19	0.543803	0.343194	0.361271	0.466999	0.107003	0.353166

TABLE 2. BASIC VARIABLES OF LOCAL SPHERICAL OPTIMUM FIGURES

b (rad)	c (rad)	d (rad)	e (rad)	g (rad)	h (rad)	h (deg)	r (deg)
0.552508	0.365712	0.415116	0.484367	0.127998	0.349066	20	70
0.494433	0.288458	0.335705	0.447879	0.107254	0.523599	30	60
0.420594	0.226806	0.266654	0.392349	0.088153	0.698132	40	50
0.379911	0.201726	0.236141	0.358045	0.079608	0.785398	45	45
0.337494	0.179884	0.207889	0.319948	0.071759	0.872665	50	40
0.252905	0.135694	0.153570	0.242150	0.055149	1.047198	60	30
0.168511	0.090798	0.101212	0.162422	0.037358	1.221730	70	20
0.084209	0.045523	0.050280	0.081483	0.018869	1.396263	80	10

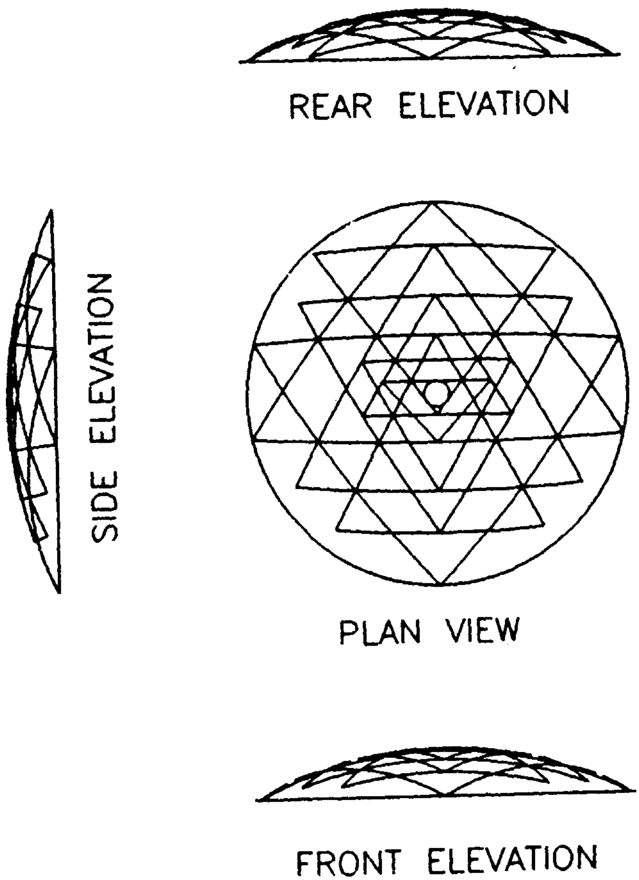


Fig.4 SPH. FIGURE FOR CONSTRAINTS 1,2,4,5,10,19

All possible combinations from the constraints formulated are investigated. The basic variables obtained for some combinations are given in Table 1. Different views of the figure with constraints 1, 2, 4, 5, 10, 19 are shown in Fig. 3.

Optimum Figures

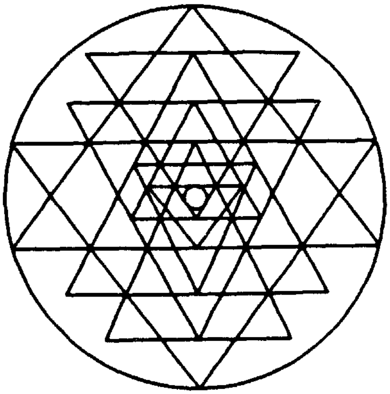
It can be seen from the results of the constrained figures that figures satisfying any 6 specified constraints can be obtained. Any given figure which may not be satisfying some essential conditions can be improved by eliminating the errors. The corrected figure with minimum changes from the given figure may be called an optimum figure.

Let L_1, L_2, ..., L_n be the finite number of given lengths used to draw the reference figure. The corresponding values H_1, H_2, ... H_n of the optimum figure can be computed in terms of of the basic variables of the present study. A measure of deviation of the optimum figure from the reference figure may be defined as the sum of squares of the deviations of the respective dimensions as follows.

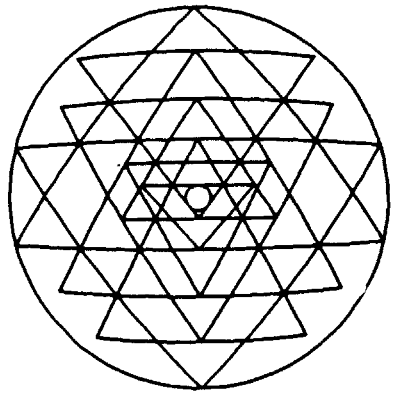
$$S = \sum_{i=1}^{n} (H_i - L_i)^2 \tag{5.1}$$

The values of the reference lengths L_1, L_2, ..., L_{10} representing the arcs $P_{10}P_9$, P_9P_8, ..., P_1P_0 are chosen as 6, 6, 5, 3, 3, 4, 3, 6, 6, 6 units corresponding to the tantra plane figure. The corresponding arcs H_1, H_2, ..., H_n are related to the variables of the present formulation as follows.

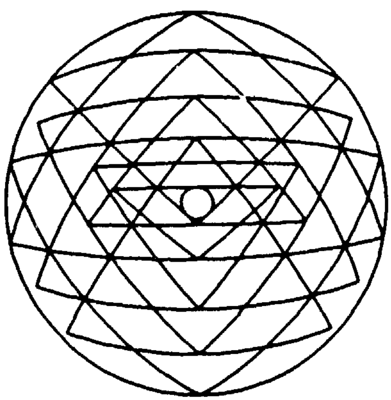
$$
\begin{aligned}
H_1 &= r - d - e & H_6 &= d + g - U_7 \\
H_2 &= e - v_8 & H_7 &= c - g \\
H_3 &= v_8 & H_8 &= v_9 \\
H_4 &= v_{12} & H_9 &= b - v_9 \\
H_5 &= U_7 - v_{12} & H_{10} &= r - b + c
\end{aligned} \tag{5.2}
$$



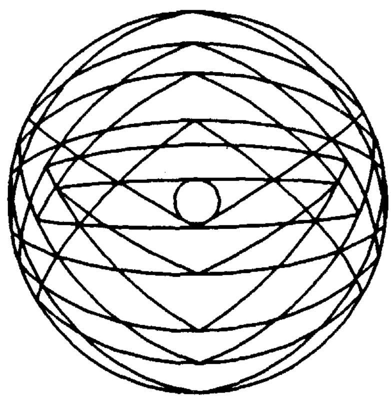
(a) ALTITUDE h = 80 deg.



(b) ALTITUDE h = 60 deg



(c) ALTITUDE h = 40 deg.



(d) ALTITUDE h = 20 deg.

Fig. 5 SPHERICAL LOCAL OPTIMUM FIGURES

The above spherical intercepts in radians are are scaled to thge radius of 24 units of the tantra figure by multiplying with a factor of $\frac{24}{r}$

The problem of finding the optimum figure is to find a set of values of the basic variables b, c, d, e, g and h such that the objective function S given in eq (5.1) is a minimum subject to the following equality and inequality constraints.

$$F_1 = 0, \; F_2 = 0, \; F_8 > 0 \;\text{ and }\; F_9 > 0 \qquad (5.3)$$

The problem is solved by using the methods of general nonlinear optimization.[15] It was observed that different sets of initial values tried for the basic variables converged to different solutions. Hence a systematic search was conducted by finding a local optimum solution for each selected value of "h" with b, c, d, e and g as the 5 basic variables. Values of "h" within the range 0 to $\frac{\pi}{2}$ at intervals of 0.01 radians were tried. The global optimum was obtained when the value of "h" was close to $\frac{\pi}{2}$. This may be due to the reference lengths being taken from the tantra plane figure.

Optimum solutions of basic variables of some selected values of "h" are shown in Table 5. Plan views of some local optimum solutions are shown in Fig. 4.

Reduction to Plane Form

The formulation of spherical form of earlier sections is simplified to obtain the formulation of plane form of *Sriyantra* where the area occupied by the triangular complex is very small compared to the total area of the sphere. The length P_0P_{10} of the arc of symmetry (Fig. 2.1) is given by 2Rr. The length of the P_0P_{10} forming the diameter of the circum circle is given by $2R_C$. The length of the chord is equated to the length of the arc to reduce the spherical formulation to plane formulation as follows.

$$R_C = R\, r \qquad (6.1)$$

Substituting for R_C from eq (2.1) and simplifying one gets

$$\sin (r) = r \qquad (6.2)$$

The following two relationship are also used which are valid when $r << 1$.

$$\tan (r) = r \text{ and } \cos (r) = 1 - \frac{r^2}{2} \qquad (6.3)$$

Similar substitutions are made for all the other variables representing the lengths of arcs having the same order of magnitude as "r". When terms of different orders of magnitudes are added, the terms of higher order are neglected. The angles between adjacent sides of triangles are retained without any change. The reduction of some typical equations is as follows.

1. Equations of the form of eq (2.2) involving arcs only.

$$r = a + b + c = d + e + f \qquad (6.5)$$

2. Replacing cosines of arcs from eq (6.3) in eq (2.3) and expanding

$$1 - \frac{x_1^2}{2} = \frac{1 - \frac{r^2}{2}}{1 - \frac{c^2}{2}} \qquad (6.6a)$$

$$x_1^2 = r^2 - c^2 \qquad (6.6)$$

3. Replacing sines and tangents using eq (6.2 and 6.3) in eq(2.5)

$$x_3 = \frac{(r - c)}{(r + d)} x_2 \qquad (6.7)$$

4. The eq(2.10) which is similar to eq(2.5) with terms transposed is reduced to

$$\frac{U_7}{(c + d)} = \frac{x_7}{x_5} \qquad (6.8)$$

5. The eq(2.12) is reduced using eq(6.2 and 6.3) as follows

$$Q_7 = \frac{V_7}{U_7} = \frac{(d + g)}{(c + d)} \frac{x_5}{x_6} \qquad (6.9)$$

6. The eq(2.13) is reduced using eq(6.2 and 6.3) as follows

$$U_7 = \frac{S_7}{Q_7 + 1} \qquad (6.10)$$

7. Replacing tangents of arcs from eq(6.3) without changing "tan (t)" in the eq(3.2b)

$$\tan(t) = \frac{(d + g - U_7)}{x_7} \qquad (6.11)$$

8. Similarly eq(3.2) reduced as follows.

$$\tan(r_T) = x_7 \frac{t}{2} \qquad (6.12)$$

9. The eq(3.3) is modified before reduction as follows

$$F_3 = \cos(d+g+v_8) - \{\cos(x_{10}) - \sin(x_{10}) \tan(x_{10})\} \qquad (6.13a)$$

$$F_3 = -\frac{(d + g + v_8)^2}{2} + \frac{3}{2} x_{10}^2 \qquad (6.13)$$

10. The eq(3.9a) is similar to eq (2.3) with terms transposed which reduces to

$$r_{16}^2 = (d+e)^2 + x_{16}^2 \qquad (6.14)$$

It was also verified[16] that these reduced equations are the same as those obtained by a direct formulation using plane geometry. Only 5 independent variable b, c, d, e and g are required to define the figure with a given value of "r". The constrained solutions with 5 conditions inclusive of the 2 essential conditions are obtained. The results for some of the combinations of the constraints are given in Table 3. Some figures drawn are shown in Fig. 5. The figure for conditions, 1, 2, 8, 9, 20 in Fig. 5c matches exactly with the figure given by Kulaichev.[7]

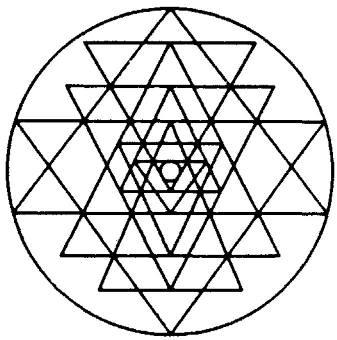
The formulation[16] of plane optimum figure can also be obtained similar to a spherical local optimum with only 5 basic variables. The intercepts H_1 to H_{10} for the plane optimum solution corresponding to reference values of 6, 6, 5, 3, 3, 4, 3, 6, 6 and 6 of the tantra plane figure are given in Table 4. The basic variables b, c, d, e and g of this optimum figure are given in Table 3. The optimum figure is shown in Fig. 5d.

TABLE 3 BASIC VARIABLES OF PLANE CONSTRAINED FIGURES (UNITS OF LENGTH: r = 1)

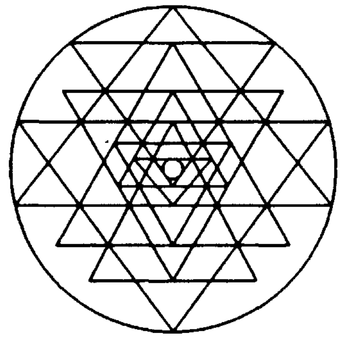
Constraints	b	c	d	e	g
1, 2, 3, 4, 8	0.463752	0.223255	0.288990	0.488181	0.106157
1, 2, 3, 10, 15	0.456449	0.236967	0.282560	0.456267	0.104822
1, 2, 4, 5, 10	0.438237	0.218371	0.269490	0.440182	0.096716
1, 2, 5, 6, 19	0.467298	0.261224	0.304553	0.471512	0.120134
1, 2, 6, 14, 19	0.468710	0.257071	0.308200	0.480582	0.121790
1, 2, 8, 9, 20	0.560659	0.279461	0.279461	0.513999	0.101410
Plane optimum	0.482391	0.261039	0.287454	0.467384	0.108463

TABLE 4. COMPARISON OF INTERCEPTS OF PLANE OPTIMUM AND TANTRA FIGURES

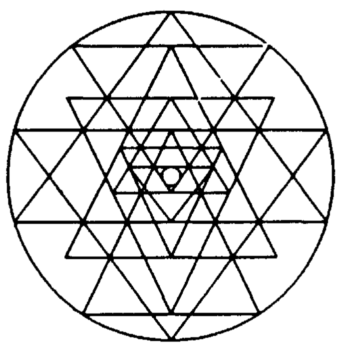
intercepts of tantra plane figure intercepts of plane Optimum figure (r = 24 Units)									
H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9	H_{10}
5.8839	6.2179	4.9993	2.7538	2.7437	4.0045	3.6618	5.8449	5.7324	6.1577
6	6	5	3	3	4	3	6	6	6



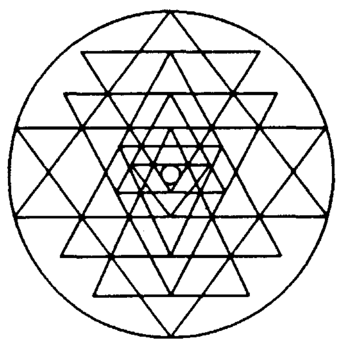
(a)CONSTR. 1 2 6 14 19



(b)CONSTR. 1 2 3 4 8



(c)CONSTR. 1 2 8 9 20



(d)PLANE OPTIMUM FIGURE

Fig.6 CONSTRAINED AND OPTIMUM PLANE FIGURES

CONCLUSIONS

A general formulation of the spherical triangle complex of *Śrīyantra* is presented. It can be seen from the accurate figures drawn using computer graphics based on the solutions obtained that the constraints formulated are satisfied. Any other property may also be added to those formulated in the present work. A condition of the form F = h - k will constrain the parameter "h" to have any specified value "k".

The incompatibility of conditions in some sets tried was indicated by some of the variables reaching values outside the permitted ranges during execution of the programms.

Optimum spherical figures with reference to tantra plane figure were obtained for a range of values of altitude "h". The value of "h" obtained was close to $\frac{\pi}{2}$ for the global optimum. Similarly any other given figure can be used as a reference. Instead of the sum of squares of deviations used in the present work, any other measure of similarity may also be used to define an optimum figure.

The validity of the formulation of the plane figures reduced from the spherical formulation shows the generality of the formulations.

Detailed dimensions of all the components of the spherical figure as a structural form of braced dome and the plane form for a *Meru* can be computed to a high degree of accuracy by specifying the required tolerances of errors.

It is interesting to observe that an unlimited number of figures with different proportions of components can be obtained which satisfy the general specifications of *Śrīyantra* from traditional literature.

The significance of *Śrīyantra* with figures of different geometrical constraints in relation to the applications needs further study.

The study has shown the effectiveness of the modern numerical methods of non linear programming applied to the study of the ancient geometrical form of *Śrīyantra*.

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